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Dp-D(p+4) in Noncommutative Yang-Mills

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Abstract

An anti-self-dual instanton solution in Yang-Mills theory on noncommutative $\mathbf{R}^4$ with an anti-self-dual noncommutative parameter is constructed. The solution is constructed by the ADHM construction and it can be treated in the framework of the IIB matrix model. In the IIB matrix model, this solution is interpreted as a system of a Dp-brane and D(p+4)-branes, with the Dp-brane dissolved in the worldvolume of the D(p+4)-branes. The solution has a parameter that characterises the size of the instanton. The zero of this parameter corresponds to the singularity of the moduli space. At this point, the solution is continuously connected to another solution which can be interpreted as a system of a Dp-brane and D(p+4)-branes, with the Dp-brane separated from the D(p+4)-branes. It is shown that even when the parameter of the solution comes to the singularity of the moduli space, the gauge field itself is non-singular. A class of multi-instanton solutions is also constructed.

1 Introduction

After the discovery of Dirichlet Branes (D-branes) [1] the nonperturbative analysis of string theory has achieved much progress. One of the key features here is that the low energy effective field theory on the worldvolume of Dp-branes describes the configuration of the D-branes in a target space, and vice-versa [2].

The Dp-D(p+4) system has attracted much interest for a number of reasons. The ground states of the Dp-D(p+4) system preserve one-fourth of the supersymmetry in superstring theory. One of the interesting features of this system is that it has descriptions from two different viewpoints. The low energy effective theory on the worldvolume of the D(p+4)-branes is a supersymmetric Yang-Mills theory, and when the Dp-branes are within the worldvolume of the D(p+4)-branes the Dp-branes are described as instantons on $\mathbf{R}^4$ transverse to the Dp-branes and within the D(p+4)-brane. On the other hand, the low energy effective field theory on the worldvolume of the Dp-branes is in the Higgs branch, and the moduli space coincides with the moduli space of instantons on $\mathbf{R}^4$ [3][4].[1] The moduli space of instantons has so-called small instanton singularities. These correspond to the instantons shrinking to zero-size, and the low energy Yang-Mills description on the D(p+4)-branes may break down. In the Dp-brane worldvolume theory, the Higgs branch meets the Coulomb branch at these small instanton singularities. The Coulomb branch describes the separation of the Dp-branes from the D(p+4)-branes in the direction transverse to the D(p+4)-branes.

The constant NS-NS B-field background in the worldvolume of D(p+4)-branes gives interesting effects to this Dp-D(p+4) system. Under the constant NS-NS B-field background the coordinates on the D(p+4)-branes become noncommutative. On the other hand, when the B-field in the $\mathbf{R}^4$ has a non-zero self-dual part (in our convention the Dp-branes are described as anti-self-dual instantons), the field theory on Dp-branes acquires a Fayet Iliopoulos term [6]. Then, the small instanton singularities in the moduli space are resolved [7], and the Coulomb branch disappears from the field theory on the worldvolume of the Dp-branes. This means that the Dp-branes are confined within the D(p+4)-branes. Since the coordinates on the D(p+4)-branes become noncommutative, the Dp-branes should be described as instantons on *noncommutative* $\mathbf{R}^4$ in the worldvolume theory on the D(p+4)-branes. This expectation is confirmed by the beautiful results in [8]; the moduli space of the field theory on the worldvolume of the Dp-branes under this background coincides with the moduli space of instantons on noncommutative $\mathbf{R}^4$.

Recently, a classical solution of the Yang-Mills theory on noncommutative $\mathbf{R}^4$ (noncommutative Yang-Mills) was found in [9]. It is a self-dual gauge field configuration on noncommutative $\mathbf{R}^4$ with a self-dual noncommutative parameter. This solution can be

[1] Here we are considering the classical moduli space. For the discussions on possible quantum corrections, see [5].

realized in the M(atrix) model [10][11] and the IIB matrix model [12][13], and then it coincides with a solution which has been studied before [14]. In these matrix models, this solution is interpreted as a system of Dp-branes and D(p+4)-branes separated from each other.

Motivated by [9], in this article we construct a new [2] $U(2)$ anti-self-dual instanton solution on noncommutative $\mathbf{R}^4$ with an anti-self-dual noncommutative parameter.[3] The construction of the anti-self-dual gauge configuration on $\mathbf{R}^4$ with anti-self-dual noncommutativity is technically the same as the construction of a self-dual gauge configuration on $\mathbf{R}^4$ with self-dual noncommutativity. The solution is constructed by the ADHM construction, and can be treated in the framework of the IIB matrix model. In the IIB matrix model, this solution can be interpreted as a system of a Dp-brane and D(p+4)-branes, with the Dp-brane dissolved in the worldvolume of the D(p+4)-branes, and with constant anti-self-dual NS-NS B-field background in the worldvolume of the D(p+4)-branes. In this case, the moduli space of instantons has a small instanton singularity. When the moduli parameter of the solution comes at this singularity, the solution is continuously connected to the solution discussed in [9][14]. It is explicitly shown that even though the parameter of the solution comes to the singularity of the moduli space, the corresponding gauge configuration is non-singular. Thus, the noncommutative Yang-Mills can describe the separation of a Dp-brane off the D(p+4)-branes, with non-singular variables. This is quite remarkable compared to the commutative case, because instantons on commutative space can describe only Dp-branes within the worldvolume of D(p+4)-branes, since the field strength becomes singular at the small instanton singularity.

We also make some comments on the instanton position parameter. When the size of the instanton becomes minimal, the instanton position parameter can be identified, in the IIB matrix model, with a position of the D(-1)-brane in the direction parallel to the D3-brane worldvolume.

A class of multi-instanton solution is also constructed. The origin of the small instanton singularities can be observed directly from the field configuration.

[2] The existence of such solution itself is a rather straightforward consequence of the ADHM construction. However, the explicit expression of the small instanton limit may be non-trivial.

[3] The case of anti-self-dual gauge configuration on $\mathbf{R}^4$ with self-dual noncommutativity has already been studied in some detail (see [8],[15]-[19]).

2 Yang-Mills Theory on Noncommutative $\mathbf{R}^4$

In this section we briefly review Yang-Mills theories on noncommutative $\mathbf{R}^4$ and their appearance in the IIB matrix model with certain backgrounds.

Gauge Fields on Noncommutative $\mathbf{R}^4$

The coordinates x^μ $(\mu = 1, \cdots, 4)$ of the noncommutative $\mathbf{R}^4$ obey the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2.1}$$

where the noncommutative parameter $\theta^{\mu\nu}$ is a real constant matrix. By $SO(4)$ rotation in $\mathbf{R}^4$ one can set the components of the matrix $\theta^{\mu\nu}$ to zero, except $\theta^{12} = -\theta^{21}$ and $\theta^{34} = -\theta^{43}$. We introduce the complex coordinates by

$$z_1 = \hat{x}^2 + i\hat{x}^1, \quad z_2 = \hat{x}^4 + i\hat{x}^3 \,. \tag{2.2}$$

Their commutation relations become

$$\begin{aligned} &[z_1, \bar{z}_1] = \zeta_1, \quad [z_2, \bar{z}_2] = \zeta_2, \\ &[z_1, z_2] = [z_1, \bar{z}_2] = 0, \end{aligned} \tag{2.3}$$

where $\zeta_1 = -2\theta^{12}$ and $\zeta_2 = -2\theta^{34}$. In this article we study the case where $\theta^{\mu\nu}$ is anti-self-dual, i.e. $\theta^{12} + \theta^{34} = 0$. This means $\zeta_1 = -\zeta_2$. Further, we set $\zeta_1 > 0$. We then define

$$a_1 \equiv \sqrt{\frac{1}{\zeta_1}} z_1, \qquad a_1^\dagger \equiv \sqrt{\frac{1}{\zeta_1}} \bar{z}_1 \,, \tag{2.4}$$

$$a_2 \equiv \sqrt{\frac{1}{\zeta_1}} \bar{z}_2, \qquad a_2^\dagger \equiv \sqrt{\frac{1}{\zeta_1}} z_2 \,. \tag{2.5}$$

We realize $a^\dagger$ and a as creation and annihilation operators acting in a Fock space $\mathcal{H}$ spanned by the basis $|n_1, n_2\rangle$:

$$\begin{aligned} a_1^\dagger |n_1, n_2\rangle &= \sqrt{n_1+1}\, |n_1+1, n_2\rangle \,, \quad a_1 |n_1, n_2\rangle = \sqrt{n_1}\, |n_1-1, n_2\rangle \,, \\ a_2^\dagger |n_1, n_2\rangle &= \sqrt{n_2+1}\, |n_1, n_2+1\rangle \,, \quad a_2 |n_1, n_2\rangle = \sqrt{n_2}\, |n_1, n_2-1\rangle \,. \end{aligned} \tag{2.6}$$

The commutation relation (2.1) has automorphismes of the form $\hat{x}^\mu \mapsto \hat{x}^\mu + y^\mu$ (translation), where y^μ is a commuting real number. We denote the Lie algebra of this group by $\underline{\mathbf{g}}$. These automorphismes are generated by the unitary operator T_y

$$T_y \equiv \exp[y^\mu \hat{\partial}_\mu], \tag{2.7}$$

where we have introduced a **derivative operator** $\hat{\partial}_\mu$ by

$$\hat{\partial}_\mu \equiv iB_{\mu\nu}\hat{x}^\nu. \tag{2.8}$$

Here, $B_{\mu\nu}$ is an inverse matrix of $\theta^{\mu\nu}$. The derivative operator $\hat{\partial}_\mu$ satisfies the following commutation relations:

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = iB_{\mu\nu}. \tag{2.9}$$

From (2.9) we obtain

$$T_y\, \hat{x}^\mu\, T_y^\dagger = \hat{x}^\mu + y^\mu. \tag{2.10}$$

For any operator $\hat{O}$ we define **derivative of operator** $\hat{O}$ by the action of $\underline{\mathbf{g}}$:

$$\partial_\mu \hat{O} \equiv \lim_{\delta y^\mu \to 0} \frac{1}{\delta y^\mu} \left(T_{\delta y^\mu} \hat{O} T_{\delta y^\mu}^\dagger - \hat{O} \right) = [\hat{\partial}_\mu, \hat{O}]. \tag{2.11}$$

The action of the exterior derivative d to the operator $\hat{O}$ is defined as

$$d\hat{O} \equiv (\partial_\mu \hat{O})\, dx^\mu. \tag{2.12}$$

Here, dx^μ's are defined in the usual way, i.e. they commute with $\hat{x}^\mu$ and anti-commute among themselves: $dx^\mu dx^\nu = -dx^\nu dx^\mu$. The covariant derivative D is written as

$$D = d + A. \tag{2.13}$$

Here, $A = A_\mu dx^\mu$ is a $U(n)$ gauge field. A_μ is an $n \times n$ anti-Hermite operator-valued matrix. The field strength of A is given by

$$F \equiv D^2 = dA + A^2 \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu. \tag{2.14}$$

We consider the following Yang-Mills action

$$S = \frac{1}{4g^2} (\pi\zeta_1)^2\, \mathrm{Tr}_{\mathcal{H}} \mathrm{tr}_{U(n)}\, F_{\mu\nu} F^{\mu\nu}. \tag{2.15}$$

The action (2.15) is invariant under the following $U(n)$ gauge transformation:

$$A \to U dU^\dagger + U A U^\dagger. \tag{2.16}$$

Here, U is a unitary operator:

$$UU^\dagger = U^\dagger U = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_n\,, \tag{2.17}$$

where $\mathrm{Id}_{\mathcal{H}}$ is the identity operator acting in $\mathcal{H}$ and Id_n is the $n \times n$ identity matrix. We will also simply write this kind of identity operators as "1" , if this is not confusing. The gauge field A is called **anti-self-dual** if its field strength obeys the following equation:

$$F^+ \equiv \frac{1}{2}(F + *F) = 0, \tag{2.18}$$

where $*$ is the Hodge star.[4] Anti-self-dual gauge fields minimize the Yang-Mills action (2.15). An instanton is an anti-self-dual gauge field with finite Yang-Mills action (2.15).

One can consider a one-to-one map from operators to ordinary c-number functions on $\mathbf{R}^4$. Under this map, noncommutative operator multiplication is mapped to the so-called star product. The map from operators to ordinary functions depends on an operator ordering prescription. Here, we choose the Weyl ordering.

Let us consider Weyl ordered operator of the form

$$\hat{f}(\hat{x}) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}_W(k)\, e^{ik\hat{x}}\,, \tag{2.19}$$

where $k\hat{x} \equiv k_\mu \hat{x}^\mu$. For the operator-valued function (2.19), the corresponding **Weyl symbol** is defined by

$$f_W(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}_W(k)\, e^{ikx}\,, \tag{2.20}$$

where x^μ's are commuting coordinates of $\mathbf{R}^4$. We define Ω_W as a map from operators to corresponding Weyl symbols:

$$\Omega_W(\hat{f}(\hat{x})) = f_W(x). \tag{2.21}$$

One can show the relation $\mathrm{Tr}_{\mathcal{H}}\{\exp(ik\hat{x})\} = (\pi\zeta_1)^2\, \delta^{(4)}(k)$. We then obtain

$$(\pi\zeta_1)^2\, \mathrm{Tr}_{\mathcal{H}}\, \hat{f}(\hat{x}) = \int d^4x\, f_W(x). \tag{2.22}$$

The **star product** of functions is defined by

$$f(x) \star g(x) \equiv \Omega_W(\Omega_W^{-1}(f(x))\Omega_W^{-1}(g(x)))\,. \tag{2.23}$$

Since

$$e^{ik\hat{x}}\, e^{ik'\hat{x}} = e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k_\nu{}'}\, e^{ik\hat{x}+ik'\hat{x}}, \tag{2.24}$$

the explicit form of the star product is given by

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}} f(x)g(x')\Big|_{x'=x}\,. \tag{2.25}$$

[4]In this note we only consider the case where the metric on $\mathbf{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$.

From the definition (2.23), the star product is associative

$$(f(x) \star g(x)) \star h(x) = f(x) \star (g(x) \star h(x)). \tag{2.26}$$

We can rewrite (2.15) using the Weyl symbols

$$S = \frac{1}{4g^2} \int \mathrm{tr}_{U(n)} \, F * F. \tag{2.27}$$

In (2.27), multiplication of the fields is understood to be the star product. The **instanton number** is defined by

$$-\frac{1}{8\pi^2} \int \mathrm{tr}_{U(n)} \, FF \,, \tag{2.28}$$

and takes an integral value.

Noncommutative Yang-Mills in the IIB Matrix Model

It is sometimes convenient to treat the classical solution of noncommutative Yang-Mills theories in the framework of the IIB matrix model. In the IIB matrix model, noncommutative Yang-Mills theories appear from expansions around certain backgrounds [21][22].

The IIB matrix model was proposed as a nonperturbative formulation of type IIB superstring theory [12][13]. It is defined by the following action:

$$S = -\frac{1}{g^2} \mathrm{Tr}_{U(N)} \left(\frac{1}{4} [X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2} \bar{\Theta} \Gamma^\mu [X_\mu, \Theta] \right) \quad (\mu = 0, \cdots, 9), \tag{2.29}$$

where X_μ and Θ are $N \times N$ hermitian matrices and each component of Θ is a Majorana-Weyl spinor. The action (2.29) has the following $U(N)$ symmetry:

$$\begin{aligned} X_\mu &\rightarrow U X_\mu U^\dagger, \\ \Theta &\rightarrow U \Theta U^\dagger, \end{aligned} \tag{2.30}$$

where U is an $N \times N$ unitary matrix:

$$U U^\dagger = U^\dagger U = \mathrm{Id}_N. \tag{2.31}$$

The action (2.29) also has the following $\mathcal{N} = 2$ supersymmetry:

$$\begin{aligned} \delta^{(1)} \Theta &= \frac{i}{2} [X_\mu, X_\nu] \Gamma^{\mu\nu} \epsilon^{(1)}, \\ \delta^{(1)} X_\mu &= i \bar{\epsilon}^{(1)} \Gamma_\mu \Theta, \\ \delta^{(2)} \Theta &= \epsilon^{(2)}, \\ \delta^{(2)} X_\mu &= 0. \end{aligned} \tag{2.32}$$

Noncommutative Yang-Mills theory appears when we consider the model in a certain classical background [21][22]. This background is a solution to the classical equation of motion, and is identified with D-brane in type IIB superstring theory. The classical equation of motion of the IIB matrix model is given by

$$[X_\mu, [X_\mu, X_\nu]] = 0. \tag{2.33}$$

One class of solutions to (2.33) is given by simultaneously diagonalizable matrices, i.e. $[X_\mu, X_\nu] = 0$ for all μ, ν. However the IIB matrix model has another class of classical solutions which are interpreted as D-branes in type IIB superstring theory:

$$\begin{aligned} X_\mu &= i\hat{\partial}_\mu \otimes \mathrm{Id}_n, \\ [i\hat{\partial}_\mu, i\hat{\partial}_\nu] &= -iB_{\mu\nu}, \end{aligned} \tag{2.34}$$

where $B_{\mu\nu}$ is a constant matrix. $i\hat{\partial}_\mu$ is an infinite-dimensional matrix because if they have only finite rank, taking a trace of both sides of (2.34) results in an apparent contradiction. (2.34) is essentially the same as the one appearing in (2.9). Therefore, we define "coordinate matrices" $\hat{x}^\mu$ from the formula (2.8):

$$\hat{x}^\mu \equiv -i\theta^{\mu\nu}\hat{\partial}_\nu, \tag{2.35}$$

where $\theta^{\mu\nu}$ is an inverse matrix of $B_{\mu\nu}$. Then, their commutation relations take the same form as those in (2.1):

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{2.36}$$

We identify these infinite-dimensional matrices with operators acting in the Fock space $\mathcal{H}$. Thus, the noncommutative coordinates of $\mathbf{R}^{2d}$ appear as a classical solution of the IIB matrix model, where $2d$ is the rank of $B_{\mu\nu}$ and the dimension of the noncommutative directions. Now, let us expand the fields around this background:

$$X_\mu = i(\hat{\partial}_\mu + A_\mu) \equiv i\hat{D}_\mu\,, \tag{2.37}$$

$$X_I = \Phi_I\,. \tag{2.38}$$

Here, μ, ν are the indices of the noncommutative directions, i.e. $det\,\theta^{\mu\nu} \neq 0$ and I, J are the indices of the directions transverse to the noncommutative directions. Then, the action (2.29) becomes

$$\begin{aligned} S = & -\frac{1}{g^2}\mathrm{Tr}_{\mathcal{H}}\mathrm{tr}_{U(n)}\Big[-\frac{1}{4}(F_{\mu\nu} + iB_{\mu\nu})(F^{\mu\nu} + iB^{\mu\nu}) + \frac{1}{2}D_\mu\Phi_I D^\mu\Phi_I \\ & +\frac{1}{4}[\Phi_I, \Phi_J][\Phi_I, \Phi_J] + \frac{1}{2}\bar{\Theta}\Gamma^\mu D_\mu\Theta + \frac{1}{2}\bar{\Theta}\Gamma^I[\Phi_I, \Theta]\Big]. \end{aligned} \tag{2.39}$$

Here,

$$D_\mu \Phi_I \equiv [\hat{D}_\mu, \Phi_I] = \partial_\mu \Phi_I + [A_\mu, \Phi_I], \tag{2.40}$$
$$D_\mu \Theta \equiv [\hat{D}_\mu, \Theta] = \partial_\mu \Theta + [A_\mu, \Theta]. \tag{2.41}$$

We thus obtain a supersymmetric noncommutative Yang-Mills theory with a $U(n)$ gauge group. The gauge transformation follows from (2.30):

$$A_\mu \rightarrow U A_\mu U^\dagger + U \partial_\mu U^\dagger, \tag{2.42}$$
$$\Phi_I \rightarrow U \Phi_I U^\dagger, \tag{2.43}$$
$$\Theta \rightarrow U \Theta U^\dagger. \tag{2.44}$$

Here, U is a unitary operator:

$$UU^\dagger = U^\dagger U = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_n. \tag{2.45}$$

The transformation of the gauge field (2.42) is determined by the requirement that the form of the derivative of operators should be kept under the gauge transformation.

As described in the previous subsection, we can rewrite the above matrix multiplication using ordinary functions and the star product.

3 Anti-Self-Dual Instantons on Noncommutative $\mathbf{R}^4$ with an Anti-Self-Dual Noncommutative Parameter

In this section we first review the ADHM construction of instantons on noncommutative $\mathbf{R}^4$, and then use it to construct an anti-self-dual instanton on noncommutative $\mathbf{R}^4$ with an anti-self-dual noncommutative parameter $\theta^{\mu\nu}$.

Review of the ADHM Construction

The ADHM construction is a way to obtain instanton solutions on $\mathbf{R}^4$ from solutions of some quadratic matrix equations [20]. It was generalized to the case of noncommutative $\mathbf{R}^4$ in [8].[5] The steps in the ADHM construction of instantons on noncommutative $\mathbf{R}^4$ with noncommutative parameter $\theta^{\mu\nu}$, gauge group $U(n)$ and instanton number k is as follows:

1. Matrices (entries are c-numbers):

$$\begin{aligned} B_1, B_2 &: \quad k \times k \quad \text{complex matrices.} \\ I, J^\dagger &: \quad k \times n \quad \text{complex matrices.} \end{aligned} \tag{3.1}$$

[5]For more detailed explanations on the ADHM construction on noncommutative $\mathbf{R}^4$, see [18][19].

2. Solve the ADHM equations:

$$\mu_{\mathbf{R}} = \zeta \qquad \text{(real ADHM equation)}, \tag{3.2}$$
$$\mu_{\mathbf{C}} = 0 \qquad \text{(complex ADHM equation)}. \tag{3.3}$$

   Here $\zeta \equiv 2(\theta^{12} + \theta^{34})$ and $\mu_{\mathbf{R}}$ and $\mu_{\mathbf{C}}$ are defined by

$$\mu_{\mathbf{R}} \equiv [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \tag{3.4}$$
$$\mu_{\mathbf{C}} \equiv [B_1, B_2] + IJ. \tag{3.5}$$

3. Define $2k \times (2k+n)$ matrix $\mathcal{D}_z$:

$$\mathcal{D}_z \equiv \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix},$$
$$\tau_z \equiv (\, B_2 - z_2,\ B_1 - z_1,\ I\,),$$
$$\sigma_z^\dagger \equiv (\, -(B_1^\dagger - \bar{z}_1),\ B_2^\dagger - \bar{z}_2,\ J^\dagger\,). \tag{3.6}$$

   Here, z and $\bar{z}$ are noncommutative *operators*.

4. Look for all solutions to the equation

$$\mathcal{D}_z \Psi^{(a)} = 0 \quad (a = 1, \ldots, n), \tag{3.7}$$

   where $\Psi^{(a)}$ is a $2k+n$ dimensional vector and its entries are *operators*. Here, we impose the following normalization condition on $\Psi^{(a)}$:

$$\Psi^{(a)\dagger} \Psi^{(b)} = \delta^{ab} \mathrm{Id}_{\mathcal{H}}\,. \tag{3.8}$$

   In the following we will call these zero-eigenvalue vectors $\Psi^{(a)}$ **zero-modes**.

5. Construct a gauge field by the formula

$$A_\mu^{ab} = \Psi^{(a)\dagger} \partial_\mu \Psi^{(b)}, \tag{3.9}$$

   where a and b become indices of the $U(n)$ gauge group. Then, this gauge field is anti-self-dual.

From the gauge field (3.9), we obtain the following expression for the field strength (for a derivation, see for example [18]):

$$\begin{aligned} & F \\ = & \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger & \xi^\dagger \end{pmatrix} \begin{pmatrix} dz_1 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{\Box_z} dz_2 & -dz_1 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_2 \frac{1}{\Box_z} dz_1 & 0 \\ -dz_2 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_1 \frac{1}{\Box_z} dz_2 & dz_2 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_1 \frac{1}{\Box_z} dz_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \\ \equiv & F^-_{\mathrm{ADHM}}\,, \end{aligned} \tag{3.10}$$

where we have written

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \equiv \begin{pmatrix} & & \\ \Psi^{(1)} & \cdots & \Psi^{(n)} \\ & & \end{pmatrix}, \qquad \begin{array}{l} \psi_1 : k \times n \text{ matrix.} \\ \psi_2 : k \times n \text{ matrix.} \\ \xi \ : n \times n \text{ matrix.} \end{array} \tag{3.11}$$

In the above we have suppressed $U(n)$ gauge indices. F^{-}_{ADHM} is anti-self-dual: $F_{1\bar{1}}+F_{2\bar{2}} = 0$, $F_{12} = 0$.

There is an action of $U(k)$ that does not change the gauge field constructed by the ADHM method:

$$(B_1, B_2, I, J) \mapsto (uB_1u^{-1}, uB_2u^{-1}, uI, Ju^{-1}), \qquad u \in U(k). \tag{3.12}$$

Therefore the moduli space $\mathcal{M}_\zeta(k,n)$ of instantons on noncommutative $\mathbf{R}^4$ with noncommutative parameter $\theta^{\mu\nu}$, gauge group $U(n)$ and instanton number k is given by

$$\mathcal{M}_\zeta(k,n) = \mu_{\mathbf{R}}^{-1}(\zeta) \cap \mu_{\mathbf{C}}^{-1}(0)/U(k). \tag{3.13}$$

Here, the action of $U(k)$ is the one given in (3.12). As stated in the previous section, in this article we consider the case where $\zeta = 0$. In this case the moduli space $\mathcal{M}_\zeta(k,n)$ has so-called small instanton singularities which appear when the size of the instanton becomes zero. When $\zeta \neq 0$, the moduli space $\mathcal{M}_\zeta(k,n)$ does not have small instanton singularities [7].

In the following we will sometimes find it more convenient to work with the variable X_μ in the IIB matrix model, rather than to work with A_μ. From (2.37) and (3.9), we obtain the following simple expression for the instanton solution X_μ:

$$\begin{aligned} X_\mu &= i\hat{\partial}_\mu + iA_\mu = i\hat{\partial}_\mu + i\Psi^\dagger\hat{\partial}_\mu\Psi - i\Psi^\dagger\Psi\hat{\partial}_\mu \\ &= i\Psi^\dagger\hat{\partial}_\mu\Psi \qquad\qquad\qquad (\mu = 1, \cdots, 4). \end{aligned} \tag{3.14}$$

From (3.14) we obtain

$$[X_\mu, X_\nu] = -F^{-}_{\mu\nu\,\mathrm{ADHM}} - iB_{\mu\nu}\,, \tag{3.15}$$

where $F^{-}_{\mu\nu\,\mathrm{ADHM}}$ is given by (3.10). From (3.15) it is easily shown that the X_μ in (3.14) satisfies the classical equation of motion of the IIB matrix model (2.33). It is also easy to show that this configuration preserves one-fourth of the supersymmetry [22].

$U(2)$ One-Instanton Solution and Small Instanton Limit

Now, let us construct an instanton by the ADHM method. The simplest solution may be a $U(2)$ one-instanton solution. In this case, B_1 and B_2 are 1×1 matrices, i.e. complex

numbers. Therefore, commutaters with B_1 and B_2 automatically give zero, and a solution to the ADHM equation (3.4) is given by

$$B_1 = B_2 = 0, \quad I = (\rho\ 0), \quad J^\dagger = (0\ \rho). \tag{3.16}$$

Then, from (3.6) we obtain

$$\mathcal{D}_z = \begin{pmatrix} -z_2 & -z_1 & \rho & 0 \\ \bar{z}_1 & -\bar{z}_2 & 0 & \rho \end{pmatrix}. \tag{3.17}$$

A solution Ψ to the equation $\mathcal{D}_z \Psi = 0$ is given by

$$\begin{aligned} \Psi &= \begin{pmatrix} \Psi^{(1)} & \Psi^{(2)} \end{pmatrix}, \\ \Psi^{(1)} &= \begin{pmatrix} \rho \\ 0 \\ z_2 \\ -\bar{z}_1 \end{pmatrix} \frac{1}{\sqrt{z_1\bar{z}_1 + \bar{z}_2 z_2 + \rho^2}} = \begin{pmatrix} \rho \\ 0 \\ \sqrt{\zeta_1} a_2^\dagger \\ -\sqrt{\zeta_1} a_1^\dagger \end{pmatrix} \frac{1}{\sqrt{\zeta_1(\hat{N}+2) + \rho^2}}, \\ \Psi^{(2)} &= \begin{pmatrix} 0 \\ \rho \\ z_1 \\ \bar{z}_2 \end{pmatrix} \frac{1}{\sqrt{\bar{z}_1 z_1 + z_2\bar{z}_2 + \rho^2}} = \begin{pmatrix} 0 \\ \rho \\ \sqrt{\zeta_1} a_1 \\ \sqrt{\zeta_1} a_2 \end{pmatrix} \frac{1}{\sqrt{\zeta_1 \hat{N} + \rho^2}}. \end{aligned} \tag{3.18}$$

Here, $\hat{N} \equiv a_1^\dagger a_1 + a_2^\dagger a_2$. The zero-mode Ψ is normalized as in (3.8):

$$\Psi^\dagger \Psi = \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & 0 \\ 0 & \mathrm{Id}_{\mathcal{H}} \end{pmatrix}. \tag{3.19}$$

The gauge field is given by (3.9):

$$A_\mu(\hat{x}) = \Psi^\dagger \partial_\mu \Psi \equiv A_\mu^{(0)}(\hat{x}). \tag{3.20}$$

We can construct following classical solution of the IIB matrix model:

$$\begin{aligned} X_\mu &= i\Psi^\dagger \hat{\partial}_\mu \Psi \qquad (\mu, \nu = 1, \cdots, 4), \\ X_I &= c_I \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2 \quad (I, J = 0, 5, \cdots 9). \end{aligned} \tag{3.21}$$

This solution is interpreted as a system of (Euclidean) D3-brane with NS-NS B-field background in its worldvolume and a D(-1)-brane dissolved in the worldvolume of the D3-branes. From (3.21) we obtain

$$\begin{aligned} [X_\mu, X_\nu] &= -F^-_{\mu\nu\,\mathrm{ADHM}} - iB_{\mu\nu}, \\ [X_\mu, X_I] &= [X_I, X_J] = 0. \end{aligned} \tag{3.22}$$

Here, μ, ν are indices of the directions along the worldvolume of the D3-branes and I, J are indices of the directions transverse to the D3-branes. The explicit form of the field strength can be obtained from (3.10):

$$\begin{aligned} F^{-}_{1\bar{1}\,\mathrm{ADHM}} = -F^{-}_{2\bar{2}\,\mathrm{ADHM}} &= \begin{pmatrix} \frac{\rho^2}{(\zeta_1(\hat{N}+1)+\rho^2)(\zeta_1(\hat{N}+2)+\rho^2)} & 0 \\ 0 & -\frac{\rho^2}{\zeta_1(\hat{N}+\rho^2)(\zeta_1(\hat{N}+1)+\rho^2)} \end{pmatrix}, \\ F^{-}_{1\bar{2}\,\mathrm{ADHM}} = -F^{-\,\dagger}_{2\bar{1}\,\mathrm{ADHM}} &= \begin{pmatrix} 0 & -\frac{2\rho^2}{(\zeta_1(\hat{N}+1)+\rho^2)\sqrt{\zeta_1(\hat{N}+\rho^2)}\sqrt{\zeta_1(\hat{N}+2)+\rho^2}} \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.23)$$

From (3.23) one can observe that the parameter ρ characterises the size of the instanton.

The calculation of the instanton number reduces to the surface integral at infinity, and when the gauge group is $U(2)$, the effect of the noncommutativity vanishes there.[6] Hence the instanton number is classified by $\pi_3(U(2))$, and the configuration (3.20) has instanton number one. It is a little fun to study how the direct calculation using the field strength (3.23) leads to the instanton number one. From (3.23), we obtain

$$\begin{aligned} F_{1\bar{1}}F_{2\bar{2}} &= -\frac{1}{\zeta_1}\begin{pmatrix} \frac{s^2}{(\hat{N}+1+s)^2(\hat{N}+2+s)^2} & 0 \\ 0 & \frac{s^2}{(\hat{N}+s)^2(\hat{N}+1+s)^2} \end{pmatrix}, \\ F_{1\bar{2}}F_{2\bar{1}} &= \frac{1}{\zeta_1}\begin{pmatrix} \frac{4s^2}{(\hat{N}+s)(\hat{N}+1+s)^2(\hat{N}+2+s)} & 0 \\ 0 & 0 \end{pmatrix}, \\ F_{2\bar{1}}F_{1\bar{2}} &= \frac{1}{\zeta_1}\begin{pmatrix} 0 & 0 \\ 0 & \frac{4s^2}{(\hat{N}+s)(\hat{N}+1+s)^2(\hat{N}+2+s)} \end{pmatrix}, \end{aligned} \quad (3.24)$$

where we have introduced a dimensionless parameter $s \equiv \frac{\rho^2}{\zeta_1}$. Then the instanton number (2.28) becomes

$$\begin{aligned} & -\frac{1}{8\pi^2}\int \mathrm{tr}_{U(2)}FF \\ =\ & (\zeta_1)^2 \mathrm{Tr}_{\mathcal{H}}\mathrm{tr}_{U(2)}\left[-F_{1\bar{1}}F_{2\bar{2}} + \frac{1}{2}\left(F_{1\bar{2}}F_{2\bar{1}} + F_{2\bar{1}}F_{1\bar{2}}\right)\right] \\ =\ & \mathrm{Tr}_{\mathcal{H}}\frac{s^2}{(\hat{N}+1+s)^2}\left(\frac{1}{(\hat{N}+2+s)^2} + \frac{1}{(\hat{N}+s)^2} + \frac{4}{(\hat{N}+2+s)(\hat{N}+s)}\right) \\ =\ & \sum_{(n_1,n_2)}\frac{s^2}{(N+1+s)^2}\left(\frac{1}{(N+2+s)^2} + \frac{1}{(N+s)^2} + 2\left(\frac{1}{(N+s)} - \frac{1}{(N+2+s)}\right)\right) \\ =\ & \sum_{N=0}^{\infty}(N+1)\frac{s^2}{(N+1+s)^2}\left(\frac{2(N+s)+1}{(N+s)^2} - \frac{2(N+s+1)+1}{(N+2+s)^2}\right) \\ =\ & \sum_{N=0}^{\infty}\frac{s^2}{(N+1+s)^2}\frac{2(N+s)+1}{(N+s)^2} \end{aligned}$$

[6] When the gauge group is $U(1)$, the effect of the noncommutativity do not vanish at infinity [18].

$$
\begin{aligned}
&+\sum_{N=0}^{\infty} s^2 \left(N \frac{2(N+s)+1}{(N+s)^2(N+1+s)^2} - (N+1)\frac{2(N+s+1)+1}{(N+1+s)^2(N+2+s)^2} \right) \\
= &\sum_{N=0}^{\infty} \frac{s^2}{(N+1+s)^2} \frac{2(N+s)+1}{(N+s)^2} \\
= &\sum_{N=0}^{\infty} \frac{s^2}{(N+s)^2} - \frac{s^2}{(N+1+s)^2} = 1.
\end{aligned} \tag{3.25}
$$

Thus the instanton number is one, independent of the parameter ρ which characterizes the size of the instanton.

Now, let us consider the small instanton limit, i.e. $\rho \to 0$. The moduli space (3.13) becomes singular at $\rho = 0$. When $\rho = 0$, the zero-mode Ψ takes the following form:

$$
\Psi = \begin{pmatrix} \Psi^{(1)} & \Psi^{(2)} \end{pmatrix},
$$

$$
\Psi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ a_2^\dagger \\ -a_1^\dagger \end{pmatrix} \frac{1}{\sqrt{\hat{N}+2}}, \quad \Psi^{(2)} = \begin{pmatrix} 0 \\ |0,0\rangle\langle 0,0| \\ a_1 \frac{1}{\sqrt{\hat{N}_{\neq 0}}} \\ a_2 \frac{1}{\sqrt{\hat{N}_{\neq 0}}} \end{pmatrix}, \tag{3.26}
$$

where $\frac{1}{\sqrt{\hat{N}_{\neq 0}}}$ is defined as

$$
\frac{1}{\sqrt{\hat{N}_{\neq 0}}} \equiv \sum_{(n_1,n_2)\neq(0,0)} \frac{1}{\sqrt{n_1+n_2}} |n_1,n_2\rangle\langle n_1,n_2| . \tag{3.27}
$$

Thus when $\rho = 0$, the explicit form of the gauge field is given by

$$
A_\mu(\hat{x}) = U^\dagger \partial_\mu U + |0,0\rangle\langle 0,0| \partial_\mu |0,0\rangle\langle 0,0| , \tag{3.28}
$$

where

$$
U \equiv \frac{1}{\sqrt{\hat{N}+1}} \begin{pmatrix} a_2^\dagger & a_1 \\ -a_1^\dagger & a_2 \end{pmatrix}. \tag{3.29}
$$

U satisfies the following equations:

$$
UU^\dagger = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2, \quad U^\dagger U = \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & 0 \\ 0 & \mathrm{Id}_{\mathcal{H}} - |0,0\rangle\langle 0,0| \end{pmatrix} \equiv p. \tag{3.30}
$$

Note that p is a projection operator: $p^2 = p$, $p^\dagger = p$. The field strength becomes

$$
F_{\mu\nu}(\hat{x}) = i(1-p)B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & |0,0\rangle\langle 0,0| \end{pmatrix} iB_{\mu\nu} \equiv F^{(0)}_{\mu\nu}(\hat{x}). \tag{3.31}
$$

Note that $\Omega_W(|0,0\rangle\langle 0,0|) = 4e^{-\frac{2}{\zeta_1}r^2}$, where $r^2 \equiv x_\mu x^\mu$. Thus the $\rho = 0$ corresponds to the "minimal size" [7] instanton. The Weyl symbol of the field strength in this case is a Gaussian function concentrated at the origin, with spreading of order $\sim \sqrt{\zeta_1}$. It is explicitly non-singular.

In terms of the IIB matrix model variable, we obtain

$$X_\mu = iU^\dagger \hat{\partial}_\mu U \equiv X_\mu^{(0)}. \tag{3.32}$$

In the above we have used the equation $\langle 0,0|\hat{\partial}_\mu|0,0\rangle = 0$.

Since $U = Up$ and $U^\dagger = pU^\dagger$, one can add new parameters to the $\rho = 0$ solution (3.32) without changing the field strength (3.31) :

$$\begin{aligned} X_\mu &= iU^\dagger\hat{\partial}_\mu U + c_\mu(1-p), \\ X_I &= C_I p + c_I(1-p), \end{aligned} \tag{3.33}$$

where we have introduced c-number parameters c_μ and c_I. c_μ's are related to the parameters of the ADHM moduli space, as we will explain in the next subsection. The field strength is unchanged by the modification from (3.21) to (3.33), in particular, it remains anti-self-dual. Hence the configuration (3.33) is still a solution of the IIB matrix model and preserves one-fourth of the supersymmetry. Since the projections p and $1-p$ are orthogonal, we can express the solution (3.33) in a block diagonal form. Taking an appropriate basis, we can write, schematically,

$$\begin{aligned} X_\mu &= \begin{pmatrix} i\hat{\partial}_\mu \otimes \mathrm{Id}_2 & \\ & c_\mu \end{pmatrix}, \\ X_I &= \begin{pmatrix} C_I \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2 & \\ & c_I \end{pmatrix}. \end{aligned} \tag{3.34}$$

Thus, the solution (3.33) is the same as that discussed in [12][14] and recently considered in [9] in the context of the noncommutative Yang-Mills theory. In the IIB matrix model, the left-upper block in (3.34) is interpreted as D3-branes, and the right-lower block is interpreted as a D(-1)-brane. The parameter set (c_μ, c_I) is interpreted as a position of the D(-1)-brane, and the parameter C_I is interpreted as a position of the D3-branes. When $C_I - c_I \neq 0$, the solution (3.33) describes the D(-1)-brane and the D3-branes separated in the direction transverse to the D3-branes.

Comments on the "Position" of the Instanton

There is an obvious extension of the solution (3.16) which follows from the translational symmetry on noncommutative $\mathbf{R}^4$:

$$B_1 = w_1,\ B_2 = w_2, \quad I = (\rho\ 0), \quad J^\dagger = (0\ \rho). \tag{3.35}$$

[7] The functional form of the gauge field depends on gauge choice.

Then, from (3.6) we obtain

$$\mathcal{D}_z = \begin{pmatrix} -(z_2 - w_2) & -(z_1 - w_1) & \rho & 0 \\ (\bar{z}_1 - \bar{w}_1) & -(\bar{z}_2 - \bar{w}_2) & 0 & \rho \end{pmatrix}. \tag{3.36}$$

Following the steps parallel to the previous subsection, we obtain a gauge field which is obtained from $A^{(0)}_\mu(\hat{x})$ in (3.20) by translation:

$$A_\mu(\hat{x}) = A^{(0)}_\mu(\hat{x} - y). \tag{3.37}$$

Here, y's are defined from

$$w_1 \equiv y^2 + iy^1, \quad w_2 \equiv y^4 + iy^3. \tag{3.38}$$

The field strength becomes

$$F_{\mu\nu}(\hat{x}) = F^{(0)}_{\mu\nu}(\hat{x} - y), \tag{3.39}$$

where $F^{(0)}_{\mu\nu}(\hat{x})$ is the one defined in (3.31). Thus the parameter y^μ can be interpreted as position of the instanton on noncommutative $\mathbf{R}^4$. This is parallel to the interpretation in the commutative case. However, on noncommutative $\mathbf{R}^4$, the notion of the position should be considered with care [23], since the translation (2.7) generates unitary gauge transformation on noncommutative $\mathbf{R}^4$:

$$A'_\mu(\hat{x}) = T_y A^{(0)}_\mu(\hat{x} - y) T^\dagger_y + T_y \partial_\mu T^\dagger_y = A^{(0)}_\mu(\hat{x}) - iB_{\mu\nu} y^\nu. \tag{3.40}$$

Thus the difference between $A'_\mu(\hat{x})$, the gauge transform of A_μ in (3.37), and $A^{(0)}_\mu(\hat{x})$ is constant. Both the translation and the constant shift of the gauge field are symmetries of the action (2.15) and (2.29). The eq. (3.40) means that these two symmetries of the action do not lead to independent moduli parameters, but they are related by the gauge transformation.

Let us study the solution in the framework of the IIB matrix model. From the ADHM data (3.35), we obtain a classical solution of the IIB matrix model:

$$X_\mu(\hat{x}) = X^{(0)}_\mu(\hat{x} - y) + c_\mu, \tag{3.41}$$

where

$$c_\mu \equiv -B_{\mu\nu} y^\nu \qquad (\mu, \nu = 1, \cdots, 4), \tag{3.42}$$

and $X^{(0)}_\mu(\hat{x})$ is defined in (3.32). Using an appropriate basis in $\mathcal{H}$, $X_\mu(\hat{x})$ can be written in a block diagonal form just like $X^{(0)}_\mu(\hat{x})$ in (3.32):

$$\begin{aligned} X_\mu(\hat{x}) &= \begin{pmatrix} (i\hat{\partial}_\mu - c_\mu) \otimes \mathrm{Id}_2 & \\ & c_\mu \end{pmatrix}, \\ X_I &= \begin{pmatrix} C_I \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2 & \\ & c_I \end{pmatrix}. \end{aligned} \tag{3.43}$$

The configuration (3.41) is gauge equivalent to (3.33). To see this, let us first consider the unitary transformation T_y. By this unitary transformation, we obtain a gauge equivalent configuration X'_μ:

$$X'_\mu(\hat{x}) = T_y \left(X^{(0)}_\mu(\hat{x} - y) + c_\mu \right) T_y^\dagger = X^{(0)}_\mu(\hat{x}) + c_\mu. \tag{3.44}$$

Next, let us consider the following unitary operator V:

$$V \equiv \left(U^\dagger T_{-y} U + (1 - p) \right) T_y. \tag{3.45}$$

By this unitary transformation V, we obtain X''_μ which is gauge equivalent to X_μ in (3.41) with $\rho = 0$:

$$X''_\mu(\hat{x}) = V X_\mu(\hat{x})\big|_{\rho=0} V^\dagger = i U^\dagger \hat{\partial}_\mu U + c_\mu (1 - p). \tag{3.46}$$

This is the expression appeared in (3.33), and there c_μ is interpreted as a position of the D(-1)-brane in the worldvolume of the D3-branes. The equation (3.42) means that the "position" parameter y^μ in the ADHM moduli is essentially equivalent to the "position" c_μ of the D(-1)-brane, in the direction parallel to the worldvolume of the D3-branes.[8]

These position parameters will enter in the gauge invariant observables like the ones considered in [23][24].

Multi-Small Instanton Solution

In general, it is quite difficult to obtain an explicit expression of the zero-modes (3.7) in the noncommutative version of the ADHM construction. The reason is as follows. When all the instantons are top on each other at the origin, then we can use the basis (2.6) of the Fock space $\mathcal{H}$. [9] However, when the instantons are "separated", there is no such convenient basis (the meaning of the word "separated" here will be made clearer shortly). However, when all the instantons become small instantons, i.e. $I = J^\dagger = 0$, we can construct an explicit k-instanton solution quite easily:

$$B_1 = \begin{pmatrix} w_1^{(1)} & 0 & \cdots & 0 \\ 0 & w_1^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & w_1^{(k)} \end{pmatrix}, \quad B_2 = \begin{pmatrix} w_2^{(1)} & 0 & \cdots & 0 \\ 0 & w_2^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & w_2^{(k)} \end{pmatrix},$$

[8]If we set $X^\mu = \hat{x}^\mu + \theta^{\mu\nu} A_\nu$ instead of (2.37), then the position parameter in the ADHM moduli exactly coincides with the position of the D(-1)-brane. Therefore, the difference in y and c in (3.42) is only a matter of the choice of the coordinate system.

[9] In [19] the explicit multi-instanton solutions with $U(1)$ gauge group is obtained. This is because in the case of the anti-self-dual instantons on self-dual noncommutativity, we can utilize a noncommutative analogue of "singular gauge", which simplifies the problem of finding zero-modes. But such a gauge choice cannot be used here. The reason may be understood from the explanation in [18], sec.4.3.

$$I = J^\dagger = 0. \tag{3.47}$$

Here $w^{(i)}$ is a parameter that expresses the position of the i-th instanton. When $w^{(i)} \neq w^{(j)}$, we will state that the i-th instanton and j-th instanton are separated.

We can construct a zero-mode corresponding to (3.47):

$$\Psi = \begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{2} \\ \left(\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & |w_1^{(1)}, w_2^{(1)}\rangle \langle 0,0| \\ \vdots & \vdots \\ 0 & |w_1^{(k)}, w_2^{(k)}\rangle \langle 0,k-1| \\ \multicolumn{2}{c}{U^k} \end{array}\right) \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\}k \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\}k \\ \}2 \ . \end{array} \tag{3.48}$$

(3.49)

Here, the numbers above and the right hand side of the matrix denote the number of the columns and the number of the lines, respectively. $|w_1^{(i)}, w_2^{(i)}\rangle$ is a coherent state:

$$\begin{aligned} z_1 |w_1^{(i)}, w_2^{(i)}\rangle &= w_1^{(i)} |w_1^{(i)}, w_2^{(i)}\rangle, \\ \bar{z}_2 |w_1^{(i)}, w_2^{(i)}\rangle &= \bar{w}_2^{(i)} |w_1^{(i)}, w_2^{(i)}\rangle, \\ \langle w_1^{(i)}, w_2^{(i)} | w_1^{(i)}, w_2^{(i)}\rangle &= 1. \end{aligned} \tag{3.50}$$

In the above, we have already chosen a gauge similar to the one in (3.46). To recognize that the zero-mode (3.48) is the correct one, we recommend the reader to check the equations $\Psi^\dagger \Psi = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2$ and $\Psi \Psi^\dagger = 1 - \mathcal{D}_z^\dagger \frac{1}{\mathcal{D}_z \mathcal{D}_z^\dagger} \mathcal{D}_z$, which are necessary conditions in the ADHM construction (see for example, [18]). To check these equations, one can utilize the following equations:

$$U^k (U^\dagger)^k = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2, \quad (U^\dagger)^k U^k = \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & 0 \\ 0 & \mathrm{Id}_{\mathcal{H}} - \sum_{n_2=0}^{k-1} |0, n_2\rangle \langle 0, n_2| \end{pmatrix}. \tag{3.51}$$

The IIB matrix variable X_μ takes the block diagonal form:

$$\begin{aligned} X_\mu &= \begin{pmatrix} i\hat{\partial}_\mu \otimes \mathrm{Id}_2 & \\ & c_\mu \end{pmatrix}, \\ X_I &= \begin{pmatrix} C_I \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2 & \\ & c_I \end{pmatrix}, \end{aligned} \tag{3.52}$$

where

$$c_\mu = \begin{pmatrix} c_\mu^{(1)} & 0 & \cdots & 0 \\ 0 & c_\mu^{(2)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c_\mu^{(k)} \end{pmatrix}, \tag{3.53}$$

$$\begin{aligned} c_\mu^{(i)} &\equiv -B_{\mu\nu} y^{\nu\,(i)}, \\ w_1^{(i)} &\equiv y^{2\,(i)} + i y^{1\,(i)}, \quad w_2^{(i)} \equiv y^{4\,(i)} + i y^{3\,(i)}, \end{aligned} \tag{3.54}$$

and c_I's are $k \times k$ matrices which commute with c_μ and among themselves. In the IIB matrix model, the $k \times k$ blocks c_μ and c_I are interpreted as the worldvolume of k D(-1)-branes. When $c^{(1)} = c^{(2)} = \cdots = c^{(k)}$, $U(k)$ symmetry enhancement occurs. Note that this $U(k)$ symmetry is the unbroken subgroup of the $U(\infty)$ unitary group acting in the Fock space $\mathcal{H}$. This symmetry enhancement is parallel to the symmetry enhancement in the solution of the ADHM equation (3.47), which is the origin of the singularities in the moduli space. It is interesting that in the noncommutative case, the origin of the singularities in the moduli space can be directly observed from the field configuration. This is owing to the fact that the field configuration is non-singular in the noncommutative case, as opposed to the commutative case.

4 Discussions

In this article we have constructed an anti-self-dual instanton solution on noncommutative $\mathbf{R}^4$ with an anti-self-dual noncommutative parameter $\theta^{\mu\nu}$. The solution is constructed by the ADHM construction, and it is discussed in the framework of the IIB matrix model. The solution has a parameter ρ that characterizes the size of the instanton. The case $\rho = 0$ corresponds to the small instanton singularity in the moduli space. It is shown that even at this small instanton singularity, the solution itself is explicitly non-singular, and takes the special form (3.32). Then the solution is continuously connected to the solution (3.33), which is interpreted as a system of separated Dp-brane and D(p+4)-branes. This is consistent with an analysis of the moduli space of field theory on the worldvolume of the Dp-brane, since in this case the Higgs branch and the Coulomb branch are connected at the small instanton singularity. It is quite remarkable that while instantons in ordinary Yang-Mills theory only describe Dp-branes within the worldvolume of D(p+4)-branes, the noncommutative Yang-Mills theory can describe the separation of Dp-branes off D(p+4)-branes.

We also observed that the instanton position parameter essentially coincides with the position of the D(-1)-brane in the IIB matrix model. It may be interesting to investigate

the appearance of the instanton position parameter in the dual supergravity side [25], in the large N super Yang-Mills/supergravity correspondence [26].

A class of multi-instanton solutions is also constructed. It is shown that he origin of symmetry enhancement in the worldvolume theory of D(-1)-branes can be observed directly from the symmetry enhancement in the IIB matrix model variable.

It will be interesting to clarify the precise relation to the sigma model analysis, like those in [27][9].

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A Smoothness of the Weyl Symbol of the Instanton Gauge Field

The components of the instanton configuration (3.20) is well defined as an operator acting in the Fock space $\mathcal{H}$, and free from divergences. However, one may wonder whether the configuration is non-singular as a function on $\mathbf{R}^4$ after the Weyl map (2.21). In this appendix, we explain how to show the smoothness of the Weyl symbol of the instanton gauge field (3.20). Hereafter we set $\zeta_1 = 1$ for simplicity, but the conclusion is the same for general ζ_1.

Let us first recall that the Weyl symbol of the projection operator $|n_1, n_2\rangle \langle n_1, n_2|$ is given by [28]

$$\Omega_W(|n_1, n_2\rangle \langle n_1, n_2|) = 2^2(-1)^{n_1+n_2} L_{n_1}(4r_1^2) L_{n_2}(4r_2^2) e^{-2r^2}, \tag{A.1}$$

where

$$\begin{aligned} r \ &\equiv \sqrt{(r_1)^2 + (r_2)^2}, \qquad r_1 \equiv \sqrt{(x^1)^2 + (x^2)^2}, \\ &\qquad\qquad\qquad\qquad\quad\; r_2 \equiv \sqrt{(x^3)^2 + (x^4)^2}, \end{aligned} \tag{A.2}$$

and $L_n(x)$ is the n-th Laguerre polynomial:

$$L_n(x) \equiv \frac{e^x}{n!} \frac{d^n}{dx^n} x^n e^{-x}. \tag{A.3}$$

Since for finite n_1 and n_2 the Weyl symbol (A.1) is finite and infinite times differentiable, potential dangerous of singularity only comes from the infinite summation over n_1 or n_2. This means that for the discussion on the smoothness of the Weyl symbols, we only need to study the region $n_1 + n_2 \equiv N \geq N_c$ for some large fixed integer N_c.

Let us write $A_\mu = A_\mu^a t^a$, where t^a's are generators of the $U(2)$ gauge group. From direct calculation, it can be shown that the gauge configuration (3.20) can be written in the form

$$A_\mu^a = \hat{f}^{(1)} \hat{P}^{(1)} + \hat{f}^{(2)} \hat{P}^{(3)}, \tag{A.4}$$

where $\hat{P}^{(1)}$ and $\hat{P}^{(3)}$ are polynomials in $\hat{x}$, and of order equal or less than one and three, respectively. Furthermore, $\hat{f}^{(1)}$ and $\hat{f}^{(2)}$ can be written as

$$\begin{aligned} \hat{f}^{(1)} &= \sum_{(n_1,n_2)} f^{(1)}_{n_1 n_2} |n_1, n_2\rangle \langle n_1, n_2|\,, \\ \hat{f}^{(2)} &= \sum_{(n_1,n_2)} f^{(2)}_{n_1 n_2} |n_1, n_2\rangle \langle n_1, n_2|\,, \end{aligned} \tag{A.5}$$

and have the asymptotic property

$$\begin{aligned} f^{(1)}_{n_1 n_2} &\rightarrow O(\frac{1}{N}) \quad (N \rightarrow \infty), \\ f^{(2)}_{n_1 n_2} &\rightarrow O(\frac{1}{N^2}) \quad (N \rightarrow \infty). \end{aligned} \tag{A.6}$$

From the explicit form (A.4), we observe that to show the smoothness of the Weyl symbol of the gauge configuration (A.4), we only need to show that $\hat{f}^{(1)}$ is two times differentiable and $\hat{f}^{(2)}$ is four times differentiable. Let us write

$$\Box^l \hat{f}^{(s)} = \sum_{(n_1,n_2)} (\Box^l f^{(s)})_{n_1 n_2} \left|n_1, n_2\right\rangle \left\langle n_1, n_2\right| \quad (s = 1, 2), \tag{A.7}$$

where $\Box \equiv \partial_\mu \partial^\mu$. Then we can check that

$$(\Box^l f^{(s)})_{n_1 n_2} \rightarrow O(\frac{1}{N^{(l+s)}}), \tag{A.8}$$

for arbitrary non-negative integer l. From (A.8) we obtain

$$\begin{aligned} \left| \Omega_W (\sum_{n_1+n_2 \geq N_c} (\Box^l f^{(s)})_{n_1 n_2} \left|n_1, n_2\right\rangle \left\langle n_1, n_2\right|) \right| &< C \sum_{n_1+n_2 \geq N_c} \frac{1}{N^{(l+s)}} \left| \Omega_W (\left|n_1, n_2\right\rangle \left\langle n_1, n_2\right|) \right| \\ &\leq C \sum_{n_1+n_2 \geq N_c} \frac{1}{N^{(l+s)}} \\ &< C' \qquad (l + s > 2), \end{aligned} \tag{A.9}$$

for some constants C and C'. In the above we have used the inequality $|L_n(x) e^{-x/2}| \leq 1$. (A.9) means that $\hat{f}^{(s)}$ is $2l$ times differentiable. Since l is an arbitrary non-negative integer, this means $\hat{f}^{(s)}$ is infinite times differentiable. Thus the Weyl symbol of the gauge configuration (A.4) is smooth.

Thus we have shown the smoothness of the gauge field (3.20). Note that even the configuration that corresponds to the singularity of the moduli space is smooth. Since even the smallest size instanton is smooth in the above case, the smoothness of the Weyl symbols of more general configurations can be expected to be shown in a similar manner, though it needs more precise arguments since we cannot write the configurations in the form (A.4) in general. We left the complete proof of the smoothness of the Weyl symbols of more general instanton configurations to the future.

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